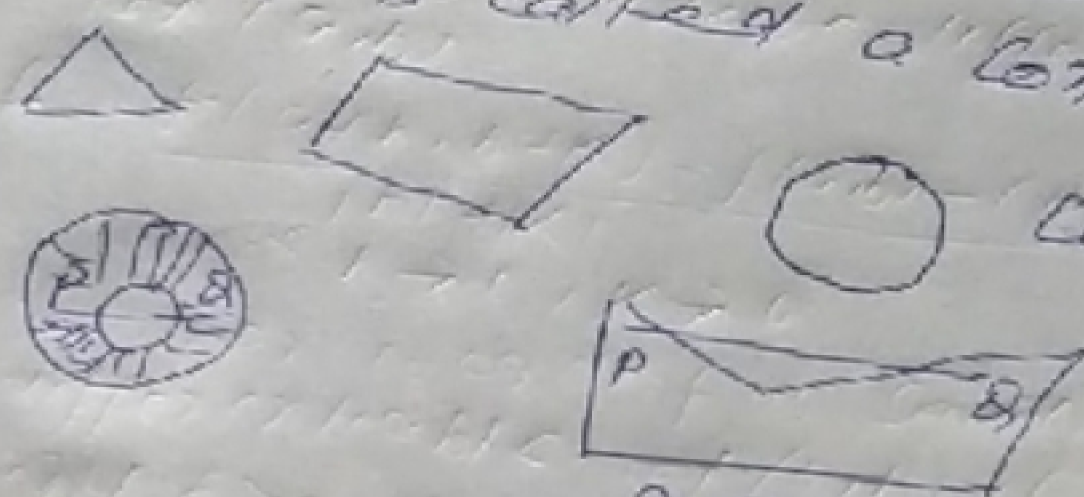
B.Sc. part-I, paper-I

(LPP) Convex set and their properties

(1) **Convex set**

Def:- Let $S \subset R^n$

If for every two points $x_1, x_2 \in S$ the line segment joining x_1 and x_2 is contained in the set S then S is called a convex set.



Convex set

non convex set

Theorem A hyperplane is a convex set

Proof:- Let $CX = K$ be a hyperplane and let x_1, x_2 be any points in it

we want to show that $C(\lambda x_1 + (1-\lambda)x_2) = K$, $0 \le \lambda \le 1$

since x_1, x_2 the hyperplane $CX = K$ we have

$$Cx_1 = K \text{ and } Cx_2 = K$$

Now $C(\lambda x_1 + (1-\lambda)x_2) = C(\lambda x_1) + C((1-\lambda)x_2)$

$= \lambda(Cx_1) + (1-\lambda)Cx_2$

$= \lambda K + (1-\lambda)K = K$

Therefore the point $\lambda x_1 + (1-\lambda)x_2$ where $0 \le \lambda \le 1$ lines in the hyper plane

Hence the hyperplane is convex set

<u>theorem</u> The intersection of two convex set is also a convex set.

proof:- Let A and B be two convex set and let $X = A \cap B$

It is requaried to prove that X is a convex set.

Let $x_1, x_2 \in X$ and let $S =$

$\{x \mid x = \lambda x_1 + (1-\lambda_2)x_2, 0 \le \lambda \le 1\}$

Now $x_1, x_2 \in X \Rightarrow x_1, x_2 \in A$

$\Rightarrow S \subset A$ ($\because A$ is convex)

Again $x_1, x_2 \in X \Rightarrow x_1, x_2 \in B$

$\Rightarrow S \subset B$ ($\because B$ is convex)

$x_1, x_2 \in X \Rightarrow S \subset A$ and $S \subset B$

$\Rightarrow S \subset A \cap B$

$\Rightarrow S \subset X$

Hence X is convex set.

theorem :- The set of all convex combinations of a finite number of linear independent vectors $v_1, v_2, v_3 \ldots v_m$ is a convex set

proof :- let $S = \{v \mid v = \sum_1^m \lambda_i v_i, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1\}$

we have to prove that S is a convex set. Let v' and $v'' \in S$

Thus to prove that S is a convex set we need to prove that

$$\lambda v' + (1-\lambda) v'' \in S$$

Now since $v' \in S$ therefor

$v' = \sum_{i=1}^m \lambda_i' v_i$ where $\lambda_i' \geq 0$ and $\sum_{i=1}^m \lambda_i' = 1$ — (1)

Again since $v'' \in S$ therefore

$v'' = \sum_{i=1}^m \lambda_i'' v_i$ where $\lambda_i'' \geq 0$ and $\sum_{i=1}^m \lambda_i'' = 1$ — (2)

Now $v = \lambda v' + (1-\lambda) v''$; $0 \leq \lambda \leq 1$

$= \lambda \sum_{i=1}^m \lambda_i' v_i + (1-\lambda) \sum_{i=1}^m \lambda_i'' v_i$

$= \sum_{i=1}^m \{\lambda \lambda_i' + (1-\lambda) \lambda_i''\} v_i$ from (1) and (2)

$= \sum_{i=1}^m \alpha_i v_i$

since $\alpha_i = \lambda \lambda_i' + (1-\lambda) \lambda_i''$; $1, 2, 3 \ldots m$

Since $0 \le \lambda \le 1$, $\lambda_i' \ge 0$, $\lambda_i'' \ge 0$, $\lambda_i \ge 0$ for each i

Also, $\sum_{i=1}^{m} \lambda_i = \sum_{i=1}^{m} \{\lambda \lambda_i' + (1-\lambda)\lambda_i''\}$

$$= \lambda \sum_{i=1}^{m} \lambda_i' + (1-\lambda) \sum_{i=1}^{m} \lambda_i''$$

$$= \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1$$

Hence y is a convex combination of the vectors $x_1, x_2, \ldots x_m$ vertices.

Thus for each pair of point x', x'' is S, the line-segment joining them is contained in the set.

Hence S is a convex set.

Theorem: The set of all feasible solutions of a L.P. problem constitutes a convex set.

Proof:- Let F be the set of all feasible solutions of the system

$$Ax = b;\ x \ge 0$$

We need to show that every convex combination of any two feasible solutions is also a feasible solution.

If the set F of solutions has only one element then F is a convex set. Hence the theorem is true in this case.

Now let us assume that $x^{(1)}$ and $x^{(2)}$ are they two feasible solution in F so that

$$Ax^{(1)} = b;\ x^{(1)} \geq 0 \quad \text{— (1)}$$

and

$$Ax^{(2)} = b;\ x^{(2)} \geq 0 \quad \text{— (2)}$$

We now consider a new point $x(0)$ as a convex combination of $x(1)$ and $x(2)$

This implies that

$$x^{(0)} = \lambda x^{(1)} + (1-\lambda)x^{(2)},\ 0 \leq \lambda \leq 1 \quad \text{— (3)}$$

Now if we can show that $x(0)$ also belongs to F then F become convex. Thus in order to show this we must show that $x(0)$ satisfies the system of constraint

$$Ax = b;\ x \geq 0 \quad \text{— (4)}$$

Now

$$Ax^{(0)} = A[\lambda x^{(1)} + (1-\lambda)x^{(2)}]$$
$$= \lambda Ax^{(1)} + (1-\lambda)Ax^{(2)}$$
$$\equiv \lambda b + (1-\lambda)b \text{ from (1) and (2)}$$
$$= b$$

this ~~satisfies~~ shows that $x(0)$ satisfied the system of constrained (4)

Also since $0 \leq \lambda \leq 1$ and since $x^{(1)}$ and $x(2) \geq 0$ it follow from (3) then $x(0) \geq 0$

Thus $x^{(0)} \in F$ and consequently F is convex set